Path integral solution for Dirac particle in a constant electric field

2009 J. Phys. A: Math. Theor. 42 015303

(http://iopscience.iop.org/1751-8121/42/1/015303)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 015303 (12pp) doi:10.1088/1751-8113/42/1/015303

Path integral solution for Dirac particle in a constant electric field

N Boudiaf, A Merdaci and L Chetouani

Département de Physique, Faculté des Sciences Exactes, Université Mentouri, 25000 Constantine, Algeria

Received 20 June 2008, in final form 17 October 2008
Published 26 November 2008
Online at stacks.iop.org/JPhysA/42/015303

Abstract
The Green's function for a Dirac particle submitted to a constant electric field is analytically calculated by using a path integral supersymmetric approach via an auxiliary equation which was introduced. The wavefunctions have been exactly obtained.

PACS numbers: 03.65.Ca, 03.65.Db, 03.65.Pm

1. Introduction

We know, nowadays, that the supersymmetry in quantum mechanics plays a more and more important role and became a means of understanding and unification of certain physical phenomena, especially in contemporary physics. In simpler words, for what concerns us the supersymmetric formalism serves for giving a unified representation for continuous variables such as the position and the impulsion, and the discrete variables such as the fundamental entity of the physics which is the spin. Thus the dynamics of the physical system (with spin) is now described by the so-called supersymmetric pseudoclassical action where the variables are the usual variables (positions and impulsions) completed by Grassmann variables related to the spin. Let us note that the variables of Grassmann introduced by Berezin and Marinov [2] are necessary to insert the spin (discrete variable) into the path integral formalism in terms of continuous paths.

Thus, alternative formalisms to the Dirac equation with applications have been developed [1, 3–7].

However if the formalisms exist, obtaining analytical solutions remains problematic, especially when it is a question of determining the exact expression of certain propagators for some types of interactions.

We know for example for an interaction as the constant electric field that the exact solution of the (3+1)-dimensional Dirac equation presents a problem that has never been exactly solved; i.e., the corresponding wavefunctions are still absent.

According to the literature available to us, only few works were devoted to the simple interaction such as the electric field: let us quote the exact solution for the Klein–Gordon equation [8], the exact solution for the one-dimensional Dirac equation [9], and the algebraic solution for the Dirac equation with a linear scalar confining potential [10].

By considering the elementary and pedagogical problem of the relativistic Dirac particle submitted to a constant electric field, our purpose in this paper is to show through the Green's function and the formalism elaborated by Fradkin and Gitman [1] how to determine the wavefunctions.

Let us note that the problem of the interaction with a constant electromagnetic field which seems to be more general has been considered by Gitman [11]; but their treatment cannot permit us to extract the wavefunctions because the spin factor depends on the trajectories.

In this paper, we propose a different method based on the introduction of a certain auxiliary equation which permits us to simplify the calculations and especially to accurately determine the spinorial part of the propagator.

The obtaining of the exact solution of this problem is only a first step: it will allow us to explore other more complex forms of potentials and to study important physical problems such as the process of creating pairs.

The propagator related to a Dirac particle in an external electromagnetic field is the causal Green's function $S^c(x_b, x_a)$ [1]

$$(\gamma \cdot \pi_b - m)S^c(x_b, x_a) = -\delta^4(x_b - x_a), \tag{1}$$

where $\pi_\mu = (\mathrm{i}\partial_\mu - gA_\mu), g$ is the electronic charge, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]_-, [\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1), \mu, \nu = \overline{0, 3}$.

The scalar product, denoted by a dot, stands for $a \cdot b = a_\mu b^\mu$.

Using for the Dirac equation another more homogeneous shape with respect to the matrices γ by multiplying by γ^5 on both sides of (1), we get

$$(\widetilde{\gamma} \cdot \pi_b - m\gamma^5)\widetilde{S}(x_b, x_a) = \delta^4(x_b - x_a), \tag{2}$$

where $\widetilde{S} = S^c\gamma^5, \widetilde{\gamma}^\mu = \gamma^5\gamma^\mu, \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \widetilde{\gamma}^5, (\gamma^5)^2 = -1$.

The matrices $\widetilde{\gamma}^\mu$ have the same commutation relations as initial ones γ^μ; $[\gamma^n, \gamma^m]_+ = 2\eta^{nm}; n, m = \overline{0, 3}, 5; \eta^{nm} = \mathrm{diag}(1, -1, -1, -1, -1)$.

Formally, $\widetilde{S}(x_b, x_a)$ is the matrix element $\langle x_b|\widetilde{S}|x_a\rangle$ in the coordinate space of the inverse Dirac operator, which is expressed as follows,

$$\widetilde{S} = (\widetilde{\gamma}^\mu\pi_\mu - m\gamma^5)^{-1} = O^{-1}, \tag{3}$$

or again in the following exponential form,

$$\widetilde{S} = S^c\gamma^5 = \int_0^\infty \mathrm{d}e \int \exp[\mathrm{i}e(O^2 + \mathrm{i}\varepsilon) + \chi O]\,\mathrm{d}\chi, \tag{4}$$

which can be very easily verified by using the results of the following integrals $\int \mathrm{d}\chi = 0$ and $\int \mathrm{d}\chi\,\chi = 1$.

By introducing Grassmann variables, it can be shown [1] that the Green's function $\widetilde{S}$ to determine the path integral formalism has the following form,

$$\begin{aligned}\widetilde{S} = \exp\left(\mathrm{i}\widetilde{\gamma} \cdot \frac{\partial_l}{\partial\theta}\right) \int_0^\infty \mathrm{d}e \int \mathrm{d}\chi \int Dx\mathcal{D}\psi\mathbb{M}(e) \\ \times \exp\left\{\mathrm{i}\int_0^1 \left[\frac{-\dot{x}^2}{2e} - \frac{e}{2}m^2 - g\dot{x} \cdot A + \mathrm{i}eg\psi \cdot F \cdot \psi\right.\right. \\ \left.\left.+ \mathrm{i}\left(\frac{\dot{x} \cdot \psi}{e} - m\psi^5\right)\chi - \mathrm{i}\psi \cdot \dot{\psi}\right]\mathrm{d}\tau + \psi(1) \cdot \psi(0)\right\}\Bigg|_{\theta=0},\end{aligned} \tag{5}$$

where x, e and χ, ψ, θ refer, respectively, to even and odd variables with the following conditions,

$$x(0) = x_a, x(1) = x_b, \qquad \psi^n(0) + \psi^n(1) = \theta^n,$$

and

$$\mathbb{M}(e) = \int D\pi \exp\left\{\frac{\mathrm{i}}{2}\int_0^1 e\pi^2\,\mathrm{d}\tau\right\},$$

$$\mathcal{D}\psi = D\psi\left[\int_{\psi(1)+\psi(0)=0} D\psi \exp\left(\int_0^1 \psi_n\dot{\psi}^n\,\mathrm{d}\tau\right)\right]^{-1},$$

are measures.

Let us choose for the interaction which describes the constant electric field, the following four-vector potential:

$$A_0 = -Ez, \qquad A_1 = A_2 = A_3 = 0. \tag{6}$$

2. Green's function—wavefunctions

With the choice of the form of the interaction, the Green's function in the phase space becomes

$$\begin{aligned}\widetilde{S} = \exp\left(\mathrm{i}\widetilde{\gamma}\cdot\frac{\partial_l}{\partial\theta}\right)\int_0^\infty \mathrm{d}e\int\mathrm{d}\chi\int DxDp\mathcal{D}\psi\\ \times\exp\left\{\mathrm{i}\int_0^1\mathrm{d}\tau\left[p\cdot\dot{x} + \frac{e}{2}(p^2-m^2) + egp\cdot A + \frac{e}{2}g^2A^2 + \mathrm{i}eg\psi\cdot F\cdot\psi\right.\right.\\ \left.\left.\left.-\,\mathrm{i}((p+gA)\cdot\psi + m\psi^5)\right]\chi - \mathrm{i}\psi\cdot\dot{\psi}\right] + \psi_n(1)\psi^n(0)\right\}\Bigg|_{\theta=0},\end{aligned} \tag{7}$$

or else, in a more explicit way, we have to determine

$$\begin{aligned}\widetilde{S} = \exp\left(\mathrm{i}\widetilde{\gamma}\cdot\frac{\partial_l}{\partial\theta}\right)\int_0^\infty\mathrm{d}e\int\mathrm{d}\chi\int\mathcal{D}\psi^1\mathcal{D}\psi^2\mathcal{D}\psi^5\mathcal{D}\widetilde{\psi}\int Dp_0Dp_1Dp_2Dp_3DtDxDyDz\\ \times\exp\left\{\mathrm{i}\int_0^1\left[p_0\dot{t} + p_1\dot{x} + p_2\dot{y} + p_3\dot{z} + \frac{e}{2}\left(p_0^2 - p_1^2 - p_2^2 - p_3^2 - m^2\right)\right.\right.\\ +\frac{e}{2}g^2E^2z^2 - egEp_0z + \mathrm{i}gEz\tilde{W}_0\cdot\widetilde{\psi}\chi - \mathrm{i}(p\cdot\psi + m\psi^5)\chi - egE\widetilde{\psi}\cdot\sigma_2\cdot\widetilde{\psi}\\ \left.-\,\mathrm{i}\widetilde{\psi}\cdot\sigma_3\cdot\dot{\widetilde{\psi}} + \mathrm{i}\psi^1\dot{\psi}^1 + \mathrm{i}\psi^2\dot{\psi}^2 + \mathrm{i}\psi^5\dot{\psi}^5\right]\mathrm{d}\tau\\ \left.+\,\widetilde{\psi}(1)\cdot\sigma_3\cdot\widetilde{\psi}(0) - \psi^1(1)\psi^1(0) - \psi^2(1)\psi^2(0) - \psi^5(1)\psi^5(0)\right\}\Bigg|_{\theta=0},\end{aligned} \tag{8}$$

where we noted $\widetilde{\psi} = \binom{0}{\psi^3}$, $\tilde{W}_0 = (1, 0)$, $\sigma_2 = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}$ and $\sigma_3 = \begin{pmatrix}1u & 0\\ 0 & 0-1\end{pmatrix}$.

It is easy to see that the integrations over the paths t, x and y allow us to make appear Dirac functions $\delta(p)$ which express that the momenta are constants of motion

$$p_0 = \text{const}, \qquad p_1 = \text{const}, \qquad p_2 = \text{const}. \tag{9}$$

Let us integrate then over $p_3 \cdot \widetilde{S}$ is thus written as

$$\widetilde{S} = \exp\left(\mathrm{i}\widetilde{\gamma}\cdot\frac{\partial_l}{\partial\theta}\right)\int_0^\infty \mathrm{d}e\int \mathrm{d}\chi\int\frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_2}{2\pi}\iint \mathcal{D}\psi^1\mathcal{D}\psi^2\mathcal{D}\psi^5\mathcal{D}\widetilde{\psi}$$
$$\times\exp\left\{\mathrm{i}p_0(t_b-t_a)+\mathrm{i}p_1(x_b-x_a)+\mathrm{i}p_2(y_b-y_a)-\frac{\mathrm{i}e}{2}\left(p_1^2+p_2^2+m^2\right)\right.$$
$$\left.+\frac{1}{e}\tilde{W}_3\cdot\left(z_b-\frac{p_0}{gE}\right)\psi(1)\chi-\frac{1}{e}\tilde{W}_3\cdot\left(z_a-\frac{p_0}{gE}\right)\psi(0)\chi\right\}$$
$$\times\exp\left\{\mathrm{i}\int_0^1[-\mathrm{i}\widetilde{\psi}\cdot\sigma_3\cdot\dot{\widetilde{\psi}}+\mathrm{i}\psi^1\dot{\psi}^1+\mathrm{i}\psi^2\dot{\psi}^2+\mathrm{i}\psi^5\dot{\psi}^5-\mathrm{i}(p_1\psi^1+p_2\psi^2+m\psi^5)\chi]\,\mathrm{d}\tau\right.$$
$$\left.-egE\widetilde{\psi}\cdot\sigma_2\cdot\widetilde{\psi}+\widetilde{\psi}(1)\cdot\sigma_3\cdot\widetilde{\psi}(0)-\psi^1(1)\psi^1(0)-\psi^2(1)\psi^2(0)-\psi^5(1)\psi^5(0)\right\}$$
$$\times\int Dz\exp\left\{\mathrm{i}\int_0^1\mathrm{d}\tau\left[\frac{\dot{z}^2}{2e}+\frac{eg^2E^2}{2}\left(z-\frac{p_0}{gE}\right)^2\right.\right.$$
$$\left.\left.\left.+\mathrm{i}\left(z-\frac{p_0}{gE}\right)\left(gE\tilde{W}_0\cdot\widetilde{\psi}+\frac{1}{e}\tilde{W}_3\cdot\dot{\widetilde{\psi}}\right)\chi\right]\right\}\right|_{\theta=0}, \qquad (10)$$

with $\tilde{W}_3 = (0, 1)$.

In order to integrate over $\psi(\tau)$, let us first eliminate the variables θ which are in the boundary conditions by performing the following changes,

$$\widetilde{\psi}(\tau) = \widetilde{\xi}(\tau)+\frac{1}{2}\tilde{\theta}+\left(v(\tau)-\frac{v(1)+v(0)}{2}\right)\chi,$$
$$\psi^1 = \xi^1+\frac{1}{2}\theta^1,\qquad \psi^2=\xi^2+\frac{1}{2}\theta^2,\qquad \psi^5=\xi^5+\frac{1}{2}\theta^5, \qquad (11)$$

with $\widetilde{\theta} = \binom{\theta^0}{\theta^3}$,

Let us note that we introduced an auxiliary variable $v(\tau)$ which will be fixed thereafter.

With this change, the Green's function becomes

$$\widetilde{S} = \exp\left(\mathrm{i}\widetilde{\gamma}\cdot\frac{\partial_l}{\partial\theta}\right)\int_0^\infty \mathrm{d}e\int \mathrm{d}\chi\int\frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_2}{2\pi}\iint \mathcal{D}\xi^1\mathcal{D}\xi^2\mathcal{D}\xi^5\mathcal{D}\widetilde{\xi}$$
$$\times\exp\left\{\mathrm{i}p_0(t_b-t_a)+\mathrm{i}p_1(x_b-x_a)+\mathrm{i}p_2(y_b-y_a)-\frac{\mathrm{i}e}{2}\left(p_1^2+p_2^2+m^2\right)\right.$$
$$-\frac{1}{4}\mathrm{i}egE\tilde{\theta}\cdot\sigma_2\cdot\tilde{\theta}-\mathrm{i}egE\widetilde{\xi}(\tau)\cdot\sigma_2\cdot\tilde{\theta}-2\mathrm{i}egE\left(\widetilde{\xi}(\tau)+\frac{1}{2}\tilde{\theta}\right)\cdot\sigma_2$$
$$\times\left(v(\tau)-\frac{v(1)+v(0)}{2}\right)\chi-\tilde{\theta}\cdot\sigma_3\cdot(v(1)-v(0))\chi+\frac{1}{2}\left(p_1\theta^1+p_2\theta^2+m\theta^5\right)\chi$$
$$+\int_0^1\left[\widetilde{\xi}(\tau)\cdot\sigma_3\cdot\dot{\widetilde{\xi}}(\tau)+2\left(\widetilde{\xi}(\tau)+\frac{1}{2}\tilde{\theta}\right)\cdot\sigma_3\cdot\dot{v}(\tau)\chi\right.$$
$$\left.\left.-\mathrm{i}egE\widetilde{\xi}(\tau)\cdot\sigma_2\cdot\widetilde{\xi}(\tau)-\xi^1\dot{\xi}^1-\xi^2\dot{\xi}^2-\xi^5\dot{\xi}^5+\left(p_1\xi^1+p_2\xi^2+m\xi^5\right)\chi\right]\mathrm{d}\tau\right\}$$
$$\times\int Dz\exp\left\{\mathrm{i}\int_0^1\left[\frac{\dot{z}^2}{2e}+\frac{eg^2E^2}{2}\left(z-\frac{p_0}{gE}\right)^2+\mathrm{i}gE\left(z-\frac{p_0}{gE}\right)\tilde{W}_0\cdot\left(\widetilde{\xi}(\tau)+\frac{1}{2}\tilde{\theta}\right)\chi\right.\right.$$
$$\left.\left.\left.-\frac{\mathrm{i}\dot{z}}{e}\tilde{W}_3\cdot\left(\widetilde{\xi}(\tau)+\frac{1}{2}\tilde{\theta}\right)\chi\right]\mathrm{d}\tau\right\}\right|_{\theta=0}, \qquad (12)$$

but where the new Grassmann variables have the following boundary conditions,

$$\widetilde{\xi}(1)+\widetilde{\xi}(0)=0, \qquad \xi^1(1)+\xi^1(0)=0,$$
$$\xi^2(1)+\xi^2(0)=0, \qquad \xi^5(1)+\xi^5(0)=0, \tag{13}$$

which are simpler (antiperiodical).

Let us fix now the auxiliary variable v by imposing

$$\left(\widetilde{\xi}(\tau)+\frac{1}{2}\tilde{\theta}\right)\left[2\sigma_3\cdot\dot{v}(\tau)-2\mathrm{i}egE\sigma_2\cdot\left(v(\tau)-\frac{v(1)+v(0)}{2}\right)\right.$$
$$\left.-gE\left(z-\frac{p_0}{gE}\right)\tilde{W}_0+\frac{\dot{z}}{e}\tilde{W}_3\right]\chi=0, \tag{14}$$

i.e. we choose an auxiliary variable $v(\tau)$ solution of the following differential equation:

$$\dot{v}(\tau)-\mathrm{i}egE\sigma_3\sigma_2\cdot\left(v(\tau)-\frac{v(1)+v(0)}{2}\right)-\frac{gE}{2}\left(z-\frac{p_0}{gE}\right)\sigma_3\tilde{W}_0+\frac{\dot{z}}{2e}\sigma_3\tilde{W}_3=0. \tag{15}$$

This first-order differential or auxiliary equation is easy to solve. The solution is

$$v(\tau)=\mathrm{e}^{egE\sigma_1\tau}\left[v(0)+\int_0^\tau \mathrm{e}^{-egE\sigma_1\tau'}\left(\frac{gE}{2}\left(z\left(\tau'\right)-\frac{p_0}{gE}\right)\tilde{W}_0\right.\right.$$
$$\left.\left.+\frac{\dot{z}(\tau')}{2e}\tilde{W}_3-egE\sigma_1\frac{v(1)+v(0)}{2}\right)\mathrm{d}\tau'\right]. \tag{16}$$

Let us note

$$v(1)-v(0)=\frac{\mathrm{e}^{\frac{1}{2}egE\sigma_1}}{\cosh\left(\frac{egE}{2}\right)}\int_0^1 \mathrm{e}^{-egE\sigma_1\tau}\left(\frac{gE}{2}\left(z(\tau)-\frac{p_0}{gE}\right)\tilde{W}_0+\frac{\dot{z}(\tau)}{2e}\tilde{W}_3\right)\mathrm{d}\tau, \tag{17}$$

where we make use of the fact that $\sigma_3\cdot\tilde{W}_0=\tilde{W}_0, \sigma_3\cdot\tilde{W}_3=-\tilde{W}_3$ and $\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

The Green's function takes a simplified form as follows,

$$\widetilde{S}=\exp\left(\mathrm{i}\widetilde{\gamma}\cdot\frac{\partial_l}{\partial\theta}\right)\int_0^\infty \mathrm{d}e\int\mathrm{d}\chi\int\frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_2}{2\pi}\iint\mathcal{D}\xi^1\mathcal{D}\xi^2\mathcal{D}\xi^5\mathcal{D}\widetilde{\xi}$$
$$\times\exp\left\{\mathrm{i}p_0(t_b-t_a)+\mathrm{i}p_1(x_b-x_a)+\mathrm{i}p_2(y_b-y_a)-\frac{\mathrm{i}e}{2}\left(p_1^2+p_2^2+m^2\right)\right.$$
$$-\frac{1}{4}\mathrm{i}egE\tilde{\theta}\cdot\sigma_2\cdot\tilde{\theta}-\mathrm{i}egE\widetilde{\xi}(\tau)\cdot\sigma_2\cdot\tilde{\theta}+\frac{1}{2}\left(p_1\theta^1+p_2\theta^2+m\theta^5\right)\chi$$
$$+\int_0^1[\widetilde{\xi}(\tau)\cdot\sigma_3\cdot\dot{\widetilde{\xi}}(\tau)-\mathrm{i}egE\widetilde{\xi}(\tau)\cdot\sigma_2\cdot\widetilde{\xi}(\tau)-\xi^1\dot{\xi}^1-\xi^2\dot{\xi}^2-\xi^5\dot{\xi}^5$$
$$\left.+\left(p_1\xi^1+p_2\xi^2+m\xi^5\right)\chi]\,\mathrm{d}\tau\right\}$$
$$\times\int Dz\exp\left\{\mathrm{i}\int_0^1\left[\frac{\dot{z}^2}{2e}+\frac{eg^2E^2}{2}\left(z-\frac{p_0}{gE}\right)^2+f(\tau)\left(z-\frac{p_0}{gE}\right)\right]\mathrm{d}\tau\right\}\Bigg|_{\theta=0}, \tag{18}$$

where

$$f(\tau)=\mathrm{i}\frac{gE}{2}\tilde{\theta}\cdot\sigma_3\cdot\frac{\mathrm{e}^{\frac{1}{2}egE\sigma_1}}{\cosh\left(\frac{egE}{2}\right)}\mathrm{e}^{-egE\sigma_1\tau}\tilde{W}_0\chi-\frac{gE}{2}\tilde{\theta}\cdot\sigma_2\frac{\mathrm{e}^{\frac{1}{2}egE\sigma_1}}{\cosh\left(\frac{egE}{2}\right)}\mathrm{e}^{-egE\sigma_1\tau}\tilde{W}_3\chi, \tag{19}$$

and where the integral over $z(\tau)$ paths is well known. It is equal to

$$\int Dz \exp\left[\mathrm{i}\int_0^1 \left(\frac{\dot{z}^2}{2e} + \frac{eg^2E^2}{2}\left(z - \frac{p_0}{gE}\right)^2 + f(\tau)\left(z - \frac{p_0}{gE}\right)\right)\mathrm{d}\tau\right] = \left(\frac{gE}{2\mathrm{i}\pi\sinh(egE)}\right)^{1/2}$$
$$\times \exp\left\{\frac{\mathrm{i}gE}{2\sinh(egE)}\left[\left(\left(z_b - \frac{p_0}{gE}\right)^2 + \left(z_a - \frac{p_0}{gE}\right)^2\right)\cosh(egE)\right.\right.$$
$$- 2\left(z_b - \frac{p_0}{gE}\right)\left(z_b - \frac{p_0}{gE}\right) + \frac{2\left(z_b - \frac{p_0}{gE}\right)}{gE}\int_0^1 f(\tau)\sinh(egE\tau)\,\mathrm{d}\tau$$
$$\left.\left. + \frac{2\left(z_a - \frac{p_0}{gE}\right)}{gE}\int_0^1 f(\tau)\sinh(egE(1-\tau))\,\mathrm{d}\tau\right]\right\}. \tag{20}$$

The Green's function becomes

$$\widetilde{S} = \exp\left(\mathrm{i}\widetilde{\gamma}\cdot\frac{\partial_l}{\partial\theta}\right)\int_0^\infty \mathrm{d}e\int\mathrm{d}\chi\int\frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_2}{2\pi}K^{os}(z_b, z_a)\iint\mathcal{D}\xi^1\mathcal{D}\xi^3\mathcal{D}\xi^5\mathcal{D}\widetilde{\xi}$$
$$\times \exp\left\{\mathrm{i}p_0(t_b - t_a) + \mathrm{i}p_1(x_b - x_a) + \mathrm{i}p_2(y_b - y_a) - \frac{\mathrm{i}e}{2}\left(p_1^2 + p_2^2 + m^2\right) - \frac{1}{4}\mathrm{i}egE\tilde{\theta}\cdot\sigma_2\cdot\tilde{\theta}\right.$$
$$+ \frac{1}{2}\left(p_1\theta^1 + p_2\theta^2 + m\theta^5\right)\chi - gE\frac{\left(z_b - \frac{p_0}{gE}\right)}{2\sinh(egE)}\left[\tanh\left(\frac{egE}{2}\right)\theta^0 - \theta^3\right]\chi$$
$$\left. - gE\frac{\left(z_a - \frac{p_0}{gE}\right)}{2\sinh(egE)}\left[\tanh\left(\frac{egE}{2}\right)\theta^0 + \theta^3\right]\chi\right\}\mathcal{I}(\theta)\bigg|_{\theta=0}, \tag{21}$$

with

$$K^{os}(y_b, y_a) = \frac{1}{\sqrt{1+\xi^2}}\exp\left[\frac{1}{4}\frac{1-\xi^2}{1+\xi^2}\left(\alpha^2+\beta^2\right) + \mathrm{i}\xi\frac{\alpha\beta}{1+\xi^2}\right]. \tag{22}$$

To extract the wavefunctions of the bosonic part of the Green's function $\widetilde{S}$, we can make use of the formula which is likely to bring about the parabolic and cylindric functions [8, 12]

$$K^{os}(z_b, z_a) = \frac{(gE)^{1/2}\,\mathrm{e}^{-\frac{5\mathrm{i}\pi}{4}}}{4\pi}\int_{\iota-\mathrm{i}\infty}^{\iota+\mathrm{i}\infty}\frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(\nu+1)}\,\mathrm{e}^{egE\left(\nu+\frac{1}{2}\right)}}{\sin(-\pi\nu)}$$
$$\times\left[D_\nu\left(\sqrt{2\mathrm{i}gE}\left(z_b - \frac{p_0}{gE}\right)\right)D_{-\nu-1}\left(\mathrm{i}\sqrt{2\mathrm{i}gE}\left(z_a - \frac{p_0}{gE}\right)\right)\right.$$
$$\left. + D_\nu\left(-\sqrt{2\mathrm{i}gE}\left(z_b - \frac{p_0}{gE}\right)\right)D_{-\nu-1}\left(-\mathrm{i}\sqrt{2\mathrm{i}gE}\left(z_a - \frac{p_0}{gE}\right)\right)\right]\mathrm{d}\nu, \tag{23}$$

where we have

$$-1 < \iota < 0, \qquad |\arg\xi| < \frac{\pi}{2},$$
$$\xi = \mathrm{i}\,\mathrm{e}^{-egE}, \qquad \alpha = \sqrt{2\mathrm{i}gE}\left(z_b - \frac{p_0}{gE}\right), \qquad \beta = \sqrt{2\mathrm{i}gE}\left(z_a - \frac{p_0}{gE}\right), \tag{24}$$

and the integral $\mathcal{I}(\theta)$ is given by

$$\mathcal{I}(\theta) = \iint\mathcal{D}\xi^1\mathcal{D}\xi^2\mathcal{D}\xi^5\mathcal{D}\widetilde{\xi}\exp\left[\int_0^1\left(\widetilde{\xi}(\tau)\mathcal{R}(\tau,\tau')\widetilde{\xi}(\tau') - \mathrm{i}egE\widetilde{\xi}(\tau)\cdot\sigma_2\cdot\tilde{\theta} - \xi^1\dot{\xi}^1 - \xi^2\dot{\xi}^2\right.\right.$$
$$\left.\left. - \xi^5\dot{\xi}^5 + \left(p_1\xi^1 + p_2\xi^2 + m\xi^5\right)\chi\,\mathrm{d}\tau\right)\right], \tag{25}$$

where

$$\mathcal{R}(\tau,\tau') = \sigma_3\delta'(\tau-\tau') - \mathrm{i}egE\sigma_2\delta(\tau-\tau'). \tag{26}$$

We are left to integrate over Grassmann variables.

As the integrations over ξ^1, ξ^2 and ξ^5 yield simple values $= 1$, the result of the integration is thus

$$\mathcal{I}(\theta) = \sqrt{\frac{\det\mathcal{R}}{\det\varepsilon}}\exp\left[\frac{1}{4}\mathcal{J}\cdot\left(\int_0^1\int_0^1\mathcal{R}^{-1}\left(\tau',\tau\right)\mathrm{d}\tau\,\mathrm{d}\tau'\right)\cdot\mathcal{J}\right], \tag{27}$$

with

$$\mathcal{J} = -\mathrm{i}egE\sigma_2\cdot\tilde{\theta}, \tag{28}$$

where $\mathcal{R}^{-1}(g\mid\tau,\tau')$ is the reverse of $\mathcal{R}(g\mid\tau,\tau')$, and can be considered as an operator which acts in the space of the antiperiodic functions

$$\mathcal{R}^{-1}(g\mid 1,\tau) + \mathcal{R}^{-1}(g\mid 0,\tau) = 0, \qquad \forall\,\tau\in[0,1], \tag{29}$$

which obeys the following equation [11],

$$\frac{\partial\mathcal{R}^{-1}(g\mid\tau,\tau')}{\partial\tau} - egE\sigma_1\mathcal{R}^{-1}(g\mid\tau,\tau') = \sigma_3\delta(\tau-\tau'). \tag{30}$$

Its solution is

$$\mathcal{R}^{-1}(g\mid\tau,\tau') = \frac{\mathrm{e}^{egE\sigma_1(\tau-\tau')}}{2}\left[\sigma_3\varepsilon(\tau-\tau') + \mathrm{i}\sigma_2\tanh\left(\frac{egE}{2}\right)\right]. \tag{31}$$

We can see

$$\iint\mathcal{R}^{-1}(g\mid\tau,\tau')\,\mathrm{d}\tau\,\mathrm{d}\tau' = \frac{\mathrm{i}\sigma_2}{(egE)^2}\left[egE - 2\tanh\left(\frac{egE}{2}\right)\right], \tag{32}$$

and we can make sure that

$$\sqrt{\frac{\det\mathcal{R}}{\det\varepsilon}} = \cosh\left(\frac{egE}{2}\right), \tag{33}$$

after some calculations, we find

$$\mathcal{I}(\theta) = \cosh\left(\frac{egE}{2}\right)\exp\left(-\frac{\mathrm{i}}{2}\tanh\left(\frac{egE}{2}\right)\tilde{\theta}\cdot\sigma_2\cdot\tilde{\theta} + \frac{1}{4}\mathrm{i}egE\tilde{\theta}\cdot\sigma_2\cdot\tilde{\theta}\right). \tag{34}$$

The Green's function takes the following form after a rather long series of calculations which are not complicated,

$$\begin{aligned}\widetilde{S} = {}& \exp\left(\mathrm{i}\widetilde{\gamma}\cdot\frac{\partial_l}{\partial\theta}\right)\int_0^\infty\mathrm{d}e\int\mathrm{d}\chi\int\frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_2}{2\pi}\cosh\left(\frac{egE}{2}\right)K^{os}(z_b,z_a)\\ &\times\exp\left[\mathrm{i}p_0(t_b-t_a) + \mathrm{i}p_1(x_b-x_a) + \mathrm{i}p_2(y_b-y_a) - \frac{\mathrm{i}e}{2}\left(p_1^2+p_2^2+m^2\right)\right]\\ &\times\exp\left(\theta_\mu Q^{\mu\nu}\theta_\nu + \frac{1}{2}\Upsilon_n\theta^n\chi\right)\Big|_{\theta=0},\end{aligned} \tag{35}$$

where we have put

$$\begin{aligned} Q^{\mu\nu} &= -\frac{F^{\mu\nu}}{2E}\tanh\left(\frac{egE}{2}\right),\\ \Upsilon_5 &= m,\\ \Upsilon_\mu &\equiv p_1W_1 + p_2W_2 - gE\frac{\left(z_b-\frac{p_0}{gE}\right)}{\sinh(egE)}\left[\tanh\left(\frac{egE}{2}\right)W_0 - W_3\right]\\ &\quad - gE\frac{\left(z_a-\frac{p_0}{gE}\right)}{\sinh(egE)}\left[\tanh\left(\frac{egE}{2}\right)W_0 + W_3\right],\end{aligned} \tag{36}$$

and

$$W_0 = (1, 0, 0, 0), \qquad W_1 = (0, 1, 0, 0), \qquad W_2 = (0, 0, 1, 0), \qquad W_3 = (0, 0, 0, 1).$$

After integrating over χ, the Green's function becomes

$$\widetilde{S} = \frac{-\mathrm{i}}{2}\int_0^\infty \mathrm{d}e \int \frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_2}{2\pi}\Phi(e)\cosh\left(\frac{egE}{2}\right)K^{os}(z_b, z_a) \times \exp\left[\mathrm{i}p_0(t_b - t_a) + \mathrm{i}p_1(x_b - x_a) + \mathrm{i}p_2(y_b - y_a) - \frac{\mathrm{i}e}{2}\left(p_1^2 + p_2^2 + m^2\right)\right], \tag{37}$$

where

$$\Phi(e) = \exp\left(\mathrm{i}\widetilde{\gamma}^n \frac{\partial_l}{\partial\theta^n}\right)(\mathrm{i}\Upsilon_n\theta^n)\exp(\theta_\mu Q^{\mu\nu}\theta_\nu)|_{\theta=0}. \tag{38}$$

Finally, to extract explicitly the spin factor, let us proceed to the derivation over the θ variables. By taking the operator $\frac{\partial_l}{\partial\theta^n}$ and then replacing the θ variables by the $\widetilde{\gamma}^n$ matrices, as in [13–16], the spin factor can be written as follows

$$\Phi(e) = -\gamma^5[m + \Upsilon^\mu\gamma_\mu + \mathrm{i}(\Upsilon^\alpha\gamma_\alpha)(Q^{\mu\nu}\sigma_{\mu\nu}) + 2\Upsilon_\mu Q^{\mu\nu}\gamma_\nu + \mathrm{i}m Q^{\mu\nu}\sigma_{\mu\nu} + m Q_{\mu\nu}Q^{*\mu\nu}\gamma^5], \tag{39}$$

where $Q^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\delta}_{\rho\delta}Q$ ($\epsilon^{\mu\nu\rho\delta}$ is the Lévi-Civita tensor),

or after development

$$\Phi(e) = \left(m + \Upsilon^\mu\gamma_\mu + (\Upsilon_0\gamma^3 + \Upsilon_3\gamma^0)\frac{\tanh\left(\frac{egE}{2}\right)}{1 - \tanh^2\left(\frac{egE}{2}\right)}\left(1 + \gamma^0\gamma^3\tanh\left(\frac{egE}{2}\right)\right)\right) \times \left(1 - \gamma^0\gamma^3\tanh\left(\frac{egE}{2}\right)\right), \tag{40}$$

we note that, as

$$\left(1 + \gamma^0\gamma^3\tanh\left(\frac{egE}{2}\right)\right)\left(1 - \gamma^0\gamma^3\tanh\left(\frac{egE}{2}\right)\right) = 1 - \tanh^2\left(\frac{egE}{2}\right). \tag{41}$$

The spin factor $\Phi(e)$ can be put under the following form:

$$\Phi(e) = -\gamma^5\left(m + \Upsilon^\mu\gamma_\mu + \frac{1}{2}\left(\Upsilon_0\gamma^3 + \Upsilon_3\gamma^0\right)\left(1 + \gamma^0\gamma^3\tanh\left(\frac{egE}{2}\right)\right)\sinh(egE)\right) \times \left(1 - \gamma^0\gamma^3\tanh\left(\frac{egE}{2}\right)\right). \tag{42}$$

The spin factor is rewritten as

$$\Phi(e) = -\gamma^5\left\{m + p_1\gamma^1 + p_2\gamma^2 - gE\left(z_a - \frac{p_0}{gE}\right)\left(\gamma^0 + \frac{\gamma^3}{\sinh(egE)}\right) + gE\left(z_b - \frac{p_0}{gE}\right)\frac{\gamma^3}{\sinh(egE)} - gE\left(z_a - \frac{p_0}{gE}\right)\gamma^3\tanh\left(\frac{egE}{2}\right)\right\}. \tag{43}$$

In order to integrate over the proper time e, let us first use the following relations:

$$\begin{aligned}\left(z_a - \frac{p_0}{gE}\right)K^{os}(z_b, z_a) &= \left[\left(z_b - \frac{p_0}{gE}\right)\cosh(egE) - \frac{\sinh(egE)}{\mathrm{i}gE}\frac{\partial}{\partial z_b}\right]K^{os}(z_b, z_a)\\ \left(z_b - \frac{p_0}{gE}\right)K^{os}(z_b, z_a) &= \left[\left(z_a - \frac{p_0}{gE}\right)\cosh(egE) - \frac{\sinh(egE)}{\mathrm{i}gE}\frac{\partial}{\partial z_a}\right]K^{os}(z_b, z_a).\end{aligned} \tag{44}$$

The spin factor is thus written as

$$\Phi(e) = -\gamma^5\left[m - gE\gamma^0\left(z_a - \frac{p_0}{gE}\right) + p_1\gamma^1 + p_2\gamma^2 + \mathrm{i}\gamma^3\frac{\partial}{\partial z_a}\right]\left(1 - \gamma^0\gamma^3\tanh\left(\frac{egE}{2}\right)\right), \tag{45}$$

with the relations related to parabolic functions

$$D_{-\nu-1}(\mathrm{i}\alpha) = \frac{\Gamma(-\nu)}{(2\pi)^{1/2}}\left[\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(\nu+1)}D_\nu(-\alpha) + \mathrm{e}^{+\mathrm{i}\frac{\pi}{2}(\nu+1)}D_\nu(\alpha)\right], \forall\alpha, \nu. \tag{46}$$

We can make sure that

$$D_\nu(\alpha)D_{-\nu-1}(\mathrm{i}\beta) + D_\nu(-\alpha)D_{-\nu-1}(-\mathrm{i}\beta) = \frac{\Gamma(-\nu)}{(2\pi)^{1/2}}\left[\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(\nu+1)}D_\nu(\alpha) + \mathrm{e}^{+\mathrm{i}\frac{\pi}{2}(\nu+1)}D_\nu(-\alpha)\right]$$
$$\times\,[\mathrm{e}^{\mathrm{i}\pi(\nu+1)}D_\nu(\beta) + D_\nu(-\beta)], \qquad \forall\,\alpha, \beta, \nu, \tag{47}$$

and using the matrices representations

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad S = \gamma^0\gamma^3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix},$$

with the vectors

$$\chi_{+1}^{+tr} = (1, 0, 0, 0), \quad \chi_{+1}^{-tr} = (0, 0, 0, 1), \quad \chi_{-1}^{+tr} = (0, 1, 0, 0), \quad \chi_{-1}^{-tr} = (0, 0, 1, 0),$$

which are the proper vectors of $S = \gamma^0\gamma^3$

$$S\chi_s^\epsilon = s\chi_s^\epsilon, \qquad \sum_{s=\pm1,\epsilon=\pm}\chi_s^\epsilon\chi_s^{\epsilon tr} = 1,$$

we can thus extract the wavefunctions starting from the spectral decomposition by going back to the definition $S^c = -\widetilde{S}\gamma^5$ which gives the Green's function related to the Dirac equation without the matrix γ^5.

The Green's function related to our particle is finally expressed as follows,

$$\mathbf{S}^c = \frac{-\mathrm{i}}{2}\int\frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_2}{2\pi}\mathcal{F}\int_0^\infty \mathrm{d}e\sum_{s=\pm1,\epsilon=\pm}\frac{(gE)^{1/2}}{4\pi}\,\mathrm{e}^{-5\mathrm{i}\frac{\pi}{4}}\int_{\iota-\mathrm{i}\infty}^{\iota+\mathrm{i}\infty}\frac{\mathrm{d}\nu}{\sin(-\pi\nu)}\,\mathrm{e}^{egE\left(\nu+\frac{1}{2}\right)}$$
$$\times\left\{D_\nu\left[\sqrt{2\mathrm{i}gE}\left(z_b - \frac{p_0}{gE}\right)\right] + \mathrm{e}^{+\mathrm{i}\pi(\nu+1)}D_\nu\left[-\sqrt{2\mathrm{i}gE}\left(z_b - \frac{p_0}{gE}\right)\right]\right\}$$
$$\times\left\{D_\nu\left[\sqrt{2\mathrm{i}gE}\left(z_a - \frac{p_0}{gE}\right)\right] + \mathrm{e}^{-\mathrm{i}\pi(\nu+1)}D_\nu\left[-\sqrt{2\mathrm{i}gE}\left(z_a - \frac{p_0}{gE}\right)\right]\right\}$$
$$\times\exp\left[\mathrm{i}p_0(t_b - t_a) + \mathrm{i}p_1(x_b - x_a) + \mathrm{i}p_2(y_b - y_a) - \frac{\mathrm{i}e}{2}\left(p_1^2 + p_2^2 + m^2 - \mathrm{i}gEs\right)\right]\chi_s^\epsilon\chi_s^{\epsilon tr}, \tag{48}$$

where

$$\mathcal{F} = -m - gE\gamma^0\left(z_a - \frac{p_0}{gE}\right) + p_1\gamma^1 + p_2\gamma^2 + \mathrm{i}\gamma^3\frac{\partial}{\partial z_a}. \tag{49}$$

Let us then integrate over e; the expression $\left[-\frac{\mathrm{i}}{2}\left(p_1^2 + p_2^2 + m^2 - \mathrm{i}gEs + 2\mathrm{i}gE\left(\nu + \frac{1}{2}\right)\right)\right]^{-1}$ appears at the denominator and out from the following pole,

$$\nu = -\frac{1}{2} + \mathrm{i}\frac{p_1^2 + p_2^2 + m^2 - \mathrm{i}gEs}{2gE} = -\frac{1}{2} + \frac{s}{2} + \frac{\mathrm{i}}{2}\lambda, \qquad \lambda = \frac{1}{gE}\left(p_1^2 + p_2^2 + m^2\right), \tag{50}$$

and with the integration over ν via the residues theorem, we finally obtain

$$\mathbf{S}^c = \frac{-\mathrm{i}}{2}\int \frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_2}{2\pi}\mathcal{F}\times\sum_{s=\pm1,\epsilon=\pm}\frac{\mathrm{e}^{-\frac{\pi}{2}(\lambda-is)}\,\mathrm{e}^{-\mathrm{i}\frac{5\pi}{4}}}{4\pi(gE)^{1/2}(1+\mathrm{e}^{-\pi(\lambda-is)})}$$
$$\times\left\{D_{\frac{s-1}{2}+\frac{\mathrm{i}}{2}\lambda}\left[\sqrt{2\mathrm{i}gE}\left(z_b-\frac{p_0}{gE}\right)\right]+\mathrm{e}^{+\mathrm{i}\frac{\pi}{2}(s+1+\mathrm{i}\lambda)}D_{\frac{s-1}{2}+\frac{\mathrm{i}}{2}\lambda}\left[-\sqrt{2\mathrm{i}gE}\left(z_b-\frac{p_0}{gE}\right)\right]\right\}$$
$$\times\left\{D_{\frac{s-1}{2}+\frac{\mathrm{i}}{2}\lambda}\left[\sqrt{2\mathrm{i}gE}\left(z_a-\frac{p_0}{gE}\right)\right]+\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(s+1+\mathrm{i}\lambda)}D_{\frac{s-1}{2}+\frac{\mathrm{i}}{2}\lambda}\left[-\sqrt{2\mathrm{i}gE}\left(z_a-\frac{p_0}{gE}\right)\right]\right\}$$
$$\times\exp\left[\mathrm{i}p_0(t_b-t_a)+\mathrm{i}p_1(x_b-x_a)+\mathrm{i}p_2(y_b-y_a)\right].\chi_s^\epsilon\chi_s^{\epsilon tr}. \tag{51}$$

We can, thus, extract the wavefunctions

$$\Psi^\epsilon_{p_0,p_1,p_2,s}(x,y,z;t)=\left(\frac{\mathrm{e}^{-\frac{\pi}{2}(\lambda-is)}\,\mathrm{e}^{-\mathrm{i}\frac{5\pi}{4}}}{4\pi(gE)^{1/2}(1+\mathrm{e}^{-\pi(\lambda-is)})}\right)^{\frac{1}{2}}$$
$$\times\left[m+gE\gamma^0\left(z_a-\frac{p_0}{gE}\right)-p_1\gamma^1-p_2\gamma^2-\mathrm{i}\gamma^3\frac{\partial}{\partial z_a}\right]$$
$$\times\left(D_{\frac{s-1}{2}+\frac{\mathrm{i}}{2}\lambda}\left(\sqrt{2\mathrm{i}gE}\left(z-\frac{p_0}{gE}\right)\right)+\mathrm{e}^{+\mathrm{i}\frac{\pi}{2}(s+1+\mathrm{i}\lambda)}D_{\frac{s-1}{2}+\frac{\mathrm{i}}{2}\lambda}\left(-\sqrt{2\mathrm{i}gE}\left(z-\frac{p_0}{gE}\right)\right)\right)$$
$$\times\mathrm{e}^{\mathrm{i}p_0t+\mathrm{i}p_1x+\mathrm{i}p_2y}\chi_s^\epsilon, \tag{52}$$

or

$$\Psi^+_{p_0,p_1,p_2,+1}(x,y,z;t)=\left(\frac{\mathrm{e}^{-\frac{\pi}{2}\lambda}\,\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}}{4\pi\,(gE)^{1/2}\,(1+\mathrm{e}^{-\pi(\lambda-\mathrm{i})})}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}p_0t+\mathrm{i}p_1x+\mathrm{i}p_2y}\begin{pmatrix} m\\ 0\\ gE\left(z-\frac{p_0}{gE}\right)-p_2\\ -p_1+\frac{\partial}{\partial z}\end{pmatrix}$$
$$\times\left\{D_{\frac{\mathrm{i}}{2}\lambda}\left[\sqrt{2\mathrm{i}gE}\left(z-\frac{p_0}{gE}\right)\right]+\mathrm{e}^{+\mathrm{i}\frac{\pi}{2}(2+\mathrm{i}\lambda)}D_{\frac{\mathrm{i}}{2}\lambda}\left[-\sqrt{2\mathrm{i}gE}\left(z-\frac{p_0}{gE}\right)\right]\right\}, \tag{53}$$

$$\Psi^+_{p_0,p_1,p_2,-1}(x,y,z;t)=\left(\frac{\mathrm{e}^{-\frac{\pi}{2}\lambda}\,\mathrm{e}^{-\mathrm{i}\frac{7\pi}{4}}}{4\pi\,(gE)^{1/2}\,(1+\mathrm{e}^{-\pi(\lambda+\mathrm{i})})}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}p_0t+\mathrm{i}p_1x+\mathrm{i}p_2y}\begin{pmatrix} 0\\ m\\ -p_1-\frac{\partial}{\partial z}\\ gE\left(z-\frac{p_0}{gE}\right)+p_2\end{pmatrix}$$
$$\times\left(D_{-1+\frac{\mathrm{i}}{2}\lambda}\left(\sqrt{2\mathrm{i}gE}\left(z-\frac{p_0}{gE}\right)\right)+\mathrm{e}^{-\frac{\pi\lambda}{2}}D_{-1+\frac{\mathrm{i}}{2}\lambda}\left(-\sqrt{2\mathrm{i}gE}\left(z-\frac{p_0}{gE}\right)\right)\right), \tag{54}$$

$$\Psi^-_{p_0,p_1,p_2,+1}(x,y,z;t)=\left(\frac{\mathrm{e}^{-\frac{\pi}{2}\lambda}\,\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}}{4\pi\,(gE)^{1/2}\,(1+\mathrm{e}^{-\pi(\lambda-\mathrm{i})})}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}p_0t+\mathrm{i}p_1x+\mathrm{i}p_2y}\begin{pmatrix} gE\left(z-\frac{p_0}{gE}\right)+p_2\\ p_1+\frac{\partial}{\partial z}\\ m\\ 0\end{pmatrix}$$
$$\times\left\{D_{\frac{\mathrm{i}}{2}\lambda}\left[\sqrt{2\mathrm{i}gE}\left(z-\frac{p_0}{gE}\right)\right]+\mathrm{e}^{+\mathrm{i}\frac{\pi}{2}(2+\mathrm{i}\lambda)}D_{\frac{\mathrm{i}}{2}\lambda}\left[-\sqrt{2\mathrm{i}gE}\left(z-\frac{p_0}{gE}\right)\right]\right\}, \tag{55}$$

$$\Psi^{-}_{p_0,p_1,p_2,-1}(x,y,z;t) = \left(\frac{e^{-\frac{\pi}{2}\lambda}\,e^{-i\frac{7\pi}{4}}}{4\pi\,(gE)^{1/2}\,(1+e^{-\pi(\lambda+i)})}\right)^{\frac{1}{2}} e^{ip_0t+ip_1x+ip_2y}\begin{pmatrix} p_1+\frac{\partial}{\partial z}\\ gE\left(z-\frac{p_0}{gE}\right)-p_2\\ 0\\ m\end{pmatrix}$$
$$\times\left\{D_{-1+\frac{i}{2}\lambda}\left[\sqrt{2igE}\left(z-\frac{p_0}{gE}\right)\right]+e^{-\frac{\pi\lambda}{2}}D_{-1+\frac{i}{2}\lambda}\left[-\sqrt{2igE}\left(z-\frac{p_0}{gE}\right)\right]\right\}. \quad (56)$$

The wavefunctions related to our problem have thus been calculated exactly and analytically.

This is our main result.

Finally let us say a word on the semiclassical limit. If we consider the expression of the Green's function which was calculated, it is clear that it is not easy to examine directly the semiclassical limit. However let us recall that in the nonrelativistic limit ($v \longrightarrow 0$) the Dirac equation in ordinary units

$$\left(\left(\frac{i\hbar}{c}\frac{\partial}{\partial t}-\frac{g}{c}A_0\right)\gamma^0-\left(i\hbar\frac{\partial}{\partial\vec{\mathbf{r}}}+\frac{g}{c}\vec{\mathbf{A}}\right)\vec{\gamma}-mc^2\right)\Psi=0,$$

where g is the electronic charge, c is the light velocity and Ψ is the bispinor $\Psi=\binom{\varphi}{\chi}$, the spinor $\chi \ll \varphi$, and φ verifies the following Pauli equation,

$$i\hbar\frac{\partial\varphi}{\partial t}=\hat{H}\varphi=\left[\frac{1}{2m}\left(\vec{\mathbf{p}}-\frac{g}{c}\vec{\mathbf{A}}\right)^2+gA_0-\frac{g\hbar}{2mc}\vec{\sigma}\cdot\vec{\mathbf{B}}\right]\varphi.$$

As in our case $\vec{\mathbf{B}}=\vec{0}$, $\vec{\mathbf{A}}=\vec{0}$ and $A_0=-Ez$ the spinor φ must verify the Schrödinger equation ($\Psi \longrightarrow \Psi^{\text{nonelativistic}} \equiv \varphi_{\text{sch}}$)

$$i\hbar\frac{\partial\varphi_{\text{sch}}}{\partial t}=\hat{H}_{\text{sch}}\varphi_{\text{sch}}=\left[\frac{1}{2m}\vec{\mathbf{p}}^2-gEz\right]\varphi_{\text{sch}}.$$

Thus we proceed to the calculation of the propagator related to the following Hamiltonian $\hat{H}_{\text{sch}}=\frac{1}{2m}\left(p_1^2+p_2^2+p_3^2\right)-gEz$; the result is well known and it is given by Feynman [17],

$$K_{\text{sch}}(\vec{\mathbf{r}}_b,\vec{\mathbf{r}}_a;t_b-t_a)=\iint\frac{dp_1}{2\pi}\frac{dp_2}{2\pi}\exp\frac{i}{\hbar}\left[p_1(x_b-x_a)+p_2(y_b-y_a)-\frac{(t_b-t_a)}{2m}\left(p_1^2+p_2^2\right)\right.$$
$$\left.+\frac{m}{2}\frac{(z_b-z_a)^2}{t_b-t_a}+\frac{gE}{2}(t_b-t_a)(z_b+z_a)-\frac{g^2E}{24}(t_b-t_a)^3\right]$$
$$=\iiint\frac{dp_1}{2\pi}\frac{dp_2}{2\pi}d\mathcal{E}\,e^{\frac{i}{\hbar}[p_1(x_b-x_a)+p_2(y_b-y_a)-\frac{t_b-t_a}{2m}(p_1^2+p_2^2)]}A_{\mathcal{E}}(z_b)A^*_{\mathcal{E}}(z_a)\,e^{-\frac{i}{\hbar}\mathcal{E}(t_b-t_a)}$$
$$=\iiint\frac{dp_1}{2\pi}\frac{dp_2}{2\pi}d\mathcal{E}\,\Psi^{\text{non-rel}}_{p_1,p_2,\mathcal{E}}(x_b,y_b,z_b)\Psi^{*\text{non-rel}}_{p_1,p_2,\mathcal{E}}(x_a,y_a,z_a)\,e^{-\frac{i}{\hbar}(t_b-t_a)\Omega(\mathcal{E},p_1,p_2)}.$$

We finally extract the wavefunctions and the spectrum energy [18]

$$\Psi^{\text{non-rel}}_{p_1,p_2,\mathcal{E}}(x,y,z)=e^{\frac{i}{\hbar}p_1x+\frac{i}{\hbar}p_2y}A_{\mathcal{E}}(z),\qquad \Omega(\mathcal{E},p_1,p_2)=\mathcal{E}+\frac{p_1^2+p_2^2}{2m},$$

and

$$A_{\mathcal{E}}(z)=\frac{1}{\sqrt{l\varepsilon}}\text{Ai}\left(\frac{z}{l}-\frac{\mathcal{E}}{\varepsilon}\right),$$

where $\varepsilon=\left(\frac{\hbar^2g^2E}{2}\right)^{\frac{1}{3}}$ and $l=\left(\frac{\hbar^2}{2gE^2}\right)^{\frac{1}{3}}=\frac{\varepsilon}{gE}$ are the natural units of energy and length, respectively, and $\text{Ai}(z)$ is the Airy function solution of

$$\frac{\partial^2\text{Ai}(z)}{\partial^2 z}=z\text{Ai}(z).$$

3. Conclusion

In the present paper by introducing an auxiliary equation, we gave an analytical and exact solution for a four-dimensional Dirac equation related to a relativistic half spin particle in interaction with a constant electric field.

The wavefunctions have been deduced.

References

[1] Fradkin E S and Gitman D M 1991 *Phys. Rev.* D **44** 3230
[2] Berezin F A and Marinov M S 1977 *Ann. Phys.* **104** 336
[3] Gitman D M 1997 *Nucl. Phys.* B **488** 490
[4] Brink L, Deser S, Zumino B, Di Vecchia P and Howe P 1976 *Phys. Lett.* B **64** 435
[5] Casalbuoni R 1976 *Nuovo Cimento* A **35** 115
[6] Barducci A, Casalbuoni R and Lusanna L 1976 *Nuovo Cimento* A **35** 377
[7] Brink L, Di Vecchia P and Howe P 1977 *Nucl. Phys.* B **118** 76
[8] Barut A O and Duru I H 1990 *Phys. Rev.* D **41** 1312
[9] Gonzalez-Diaz Luis and Villalba V M 2006 *Phys. Lett.* A **352** 202
[10] Ru-Keng Su and Yuhong Zhang 1984 *J. Phys. A: Math. Gen.* **17** 851
[11] Gitman D M and Zlatev S I 1997 *Phys. Rev.* D **55** 7701
[12] Erdelýi A (ed) 1953 *Higher Transcendental Functions* vol II (Bateman Manuscript Project) (New York: McGraw-Hill)
[13] Merdaci A, Boudiaf N and Chetouani L 2008 *Z. Naturforsch* **63a** 283
[14] Boudiaf N, Boudjedaa T and Chetouani L 2001 *Eur. Phys. J.* C **20** 585
[15] Boudiaf N, Boudjedaa T and Chetouani L 2001 *Eur. Phys. J.* C **22** 593
[16] Boudiaf N and Chetouani L 2003 *Phys. Scr.* **67** 369
[17] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
[18] Kleinert H 1990 *Path Integrals in quantum Mechanics, Statistics and Polymer* (Singapore: World Scientific)